Review: Limits of Functions - 10/7/16

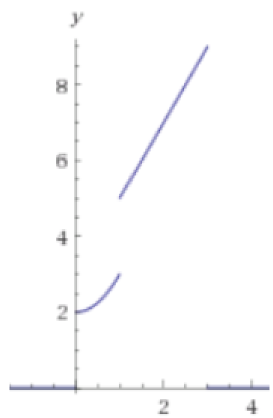
1 Right and Left Hand Limits

Definition 1.0.1 *We write* $\lim_{x\to a^-} f(x) = L$ *to mean that the function* $f(x)$ *approaches* L *as* x *approaches* a *from the left. We call this the* ***left hand limit*** *of* $f(x)$ *as* x *approaches* a.

We write $\lim_{x\to a^+} f(x) = L$ *to mean that the function* $f(x)$ *approaches* L *as* x *approaches* a *from the right. We call this the* ***right hand limit*** *of* $f(x)$ *as* x *approaches* a.

Definition 1.0.2 $\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} = \lim_{x\to a^+} = L$.

Example 1.0.3 .



Let the above graph be $f(x)$*. Then* $\lim_{x\to 1^-} f(x) = 3$*, and* $\lim_{x\to 1^+} f(x) = 5$*. Since* $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$*, the limit as* x *approaches 1 does not exist.*

For 2, $\lim_{x\to 2^-} f(x) = 7 = \lim_{x\to 2^+} f(x)$*. Thus* $\lim_{x\to 2} f(x) = 7$.

Example 1.0.4 *Let* $g(x) = \frac{x+1}{x}$*. What is* $\lim_{x\to 1^-} g(x)$*? We want to evaluate some values of* x *that are less than 1, but getting closer and closer to 1.*

x	$g(x)$
.5	3
.9	2.111
.99	2.0101
.999	2.0010

It looks like as we're getting closer and closer to 1 from the left, the values of $g(x)$ *are getting closer and closer to 2, so we can guess that* $\lim_{x\to 1^-} = 2$*. Now what about* $\lim_{x\to 1^+} g(x)$*? We do the same thing, but with values of* x *that are bigger than 1.*

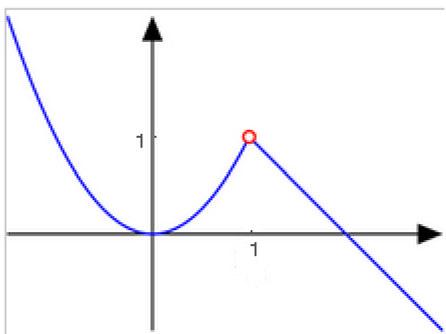
x	$g(x)$
1.5	*1.667*
1.1	*1.90909*
1.01	*1.99009*
1.001	*1.9990*

It looks like as we're getting closer and closer to 1 from the right, the values of $g(x)$ *are also getting closer and closer to 2, so we can guess that* $\lim_{x\to 1^+} = 2$*. Since we got the same values for both, we can guess that* $\lim_{x\to 1} g(x) = 2$*.*

2 Discontinuities

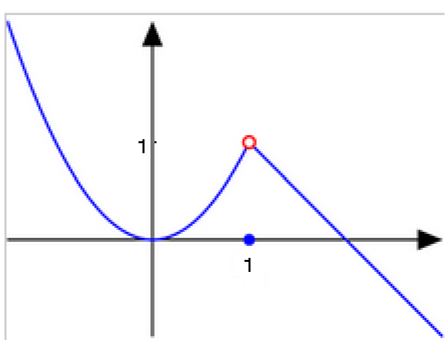
The limit $\lim_{x\to a} f(x)$ is not impacted by the value $f(a)$, only by the values $f(x)$ for the x around a.

Example 2.0.5 *Let the following function be* $f(x)$*.*



Then what is $\lim_{x\to 1^-} f(x)$*? We have* $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$*, so our overall limit is* $\lim_{x\to 1} f(x) = 1$*.*

Now let the following function be $g(x)$*.*



What is $\lim_{x\to 1^-} g(x)$*? We still have* $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = 1$*, so our overall limit is still* $\lim_{x\to 1} = 1$*.*

3 Vertical Asymptotes

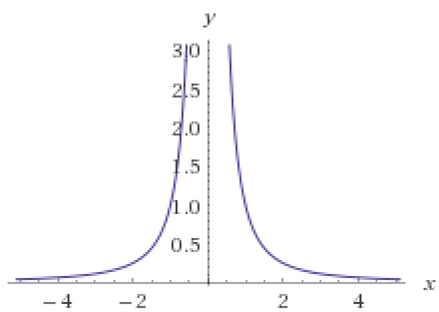
Definition 3.0.6 *We say that* $\lim_{x\to a} f(x) = \infty$ *if* $f(x)$ *can be made arbitrarily large by taking* x *values close to* a *on both sides.*

We say that $\lim_{x\to a} f(x) = -\infty$ *if* $f(x)$ *can be made arbitrarily small by taking* x *values close to* a *on both sides.*

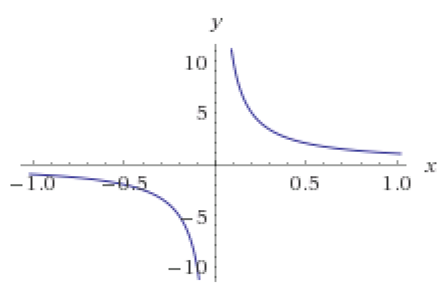
Example 3.0.7 $\lim_{x\to 0} \frac{1}{x^2}$. *Let's use both of our techniques here - estimating numerically and drawing a graph. If we estimate numerically, we can make a chart:*

x	$f(x)$
$\pm.5$	*4*
$\pm.1$	*100*
$\pm.01$	*10,000*
$\pm.001$	*1,000,000*

On either side of 0, when we plug in values of x *we just get bigger and bigger, so we guess that* $\lim_{x\to 0} f(x) = \infty$. *We could also draw the graph:*



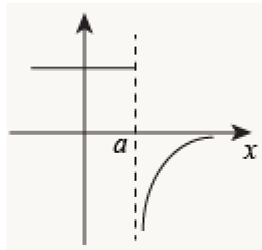
Example 3.0.8 *Let* $g(x) = \frac{1}{x}$. *Then let's look at the graph:*



$\lim_{x\to 0^-} g(x) = -\infty$ *and* $\lim_{x\to 0^+} g(x) = \infty$. *Thus the limit does not exist.*

Definition 3.0.9 *The line* $x = a$ *is a* ***vertical asymptote*** *if* $\lim_{x\to a^+}$ *or* $\lim_{x\to a^-}$ *(or both) is infinite (*∞ *of* $-\infty$*).*

Both of the above examples have vertical asymptotes at zero, even though the limit itself doesn't exist for $\frac{1}{x}$. We can also have only one side approaching infinity:



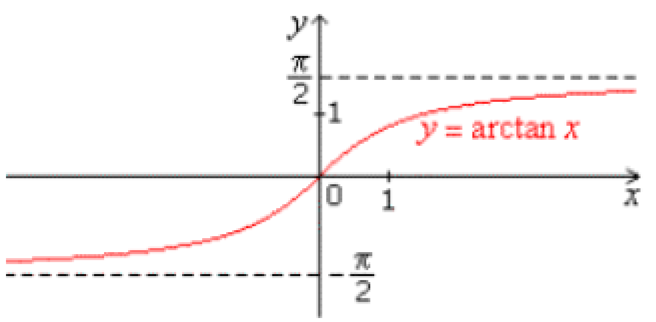
A rational function $R(x) = \frac{P(x)}{Q(x)}$ has a vertical asymptote at a if $Q(a) = 0$ AND after canceling terms with the numerator, a still gives zero in the denominator.

Example 3.0.10 *How many vertical asymptotes does $g(x) = \frac{x^2-1}{x^2-x-2}$ have? What are they? We start by seeing where the denominator is zero. $x^2 - x - 2 = (x-2)(x+1)$, so it is zero at $x = -1, 2$. Can either of these factor with the numerator? The numerator factors as $(x+1)(x-1)$, so the fraction can be written as $\frac{(x+1)(x-1)}{(x+1)(x-2)}$, so we could cancel out the $x+1$. Thus there is a hole at $x = -1$. Can we cancel the $x - 2$? No, so this is a vertical asymptote. Thus we have one vertical asymptote at $x = 2$.*

4 Horizontal Asymptotes

Definition 4.0.11 *The line $y = L$ is a horizontal asymptote of $f(x)$ if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.*

Example 4.0.12 $\lim_{x\to\infty} \arctan(x) = \pi/2$, $\lim_{x\to-\infty} \arctan(x) = -\pi/2$. *Thus* $\arctan(x)$ *has horizontal asymptotes at $\pi/2$ and $-\pi/2$.*



Say we have a function $R(x) = \frac{P(x)}{Q(x)}$.

- If the degree of P is greater that the degree of Q (e.g. $f(x) = \frac{x^3+3}{x^2-1}$), then the function does not have a horizontal asymptote.
- If the degree of P is less than the degree of Q (e.g. $g(x) = \frac{x^2+7}{x^5-4}$), then the function has a horizontal asymptote at zero.
- If the degree of P equals the degree of Q (e.g. $h(x) = \frac{3x+5}{x-7}$), then the horizontal asymptote is the ratio of the leading coefficients (here we would have a horizontal asymptote in both directions of 3).

Example 4.0.13 *What is are the vertical and horizontal asymptotes of* $f(x) = \frac{x+3}{x^2-x-12}$*? Let's start with horizontal asymptotes. The bottom grows a lot faster than the top when I plug in large numbers, so* $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$*. Thus I have a horizontal asymptote at zero. For vertical asymptotes, I can factor the bottom to give* $f(x) = \frac{x+3}{(x+3)(x-4)}$*. So is* -3 *and asymptote? No, it is a hole that can be filled in? Is 4 an asymptote? I can never get rid of the* $x-4$ *on the bottom, so* $x = 4$ *is a vertical asymptote.*

Practice Problems

1. What is $\lim_{x\to3} \frac{x+2}{x-1}$? Either estimate numerically or use a graph (or both).
2. What is $\lim_{x\to1} \frac{x^2-1}{x-1}$? Either estimate numerically or use a graph (or both). Is this an asymptote or a hole?
3. What are the horizontal and vertical asymptotes of $\frac{x-3}{x^2-4x+3}$?
4. What are the horizontal and vertical asymptotes of $\frac{x^2-1}{x+1}$?
5. What are the horizontal and vertical asymptotes of $\frac{x+2}{3x-1}$?
6. What are the horizontal and vertical asymptotes of $\frac{x^2+2x-15}{x^2+x-2}$?

Solutions

1. $\lim_{x\to3} \frac{x+2}{x-1} = 2.5$.
2. $\lim_{x\to1} \frac{x^2-1}{x-1} = 2$. This is a hole.
3. The horizontal asymptote is zero, and the vertical asymptote is $x = 1$.
4. There are no horizontal or vertical asymptotes.
5. The horizontal asymptote is $y = \frac{1}{3}$, and the vertical asymptote is $x = \frac{1}{3}$.
6. The horizontal asymptote is 1. The vertical asymptotes are $x = -2$ and $x = 1$.